Conformal scalar and torsion do not go together

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Abstract

We investigate the conformal transformation of vierbein-Einstein-Palatini (VEP) action in terms of tetrads e^I_μ and spin connection A^{IJ}_μ. The transformation of the spin connection is indeterminate off-shell unless equations of motion are satisfied. We construct the conformally invariant scalar field in the torsion-free VEP formalism. In presence of fermionic matter, torsion does not vanish, and shows up in the dynamics of conformal scalar, affecting the invariance. It is not possible to maintain conformal invariance of the scalar field equation when fermions are present.

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I. INTRODUCTION

The vierbein-Einstein-Palatini (VEP) formalism is a first-order formulation of gravity, closer in form to non-Abelian gauge theories [1, 3, 4]. This is a formulation in terms of local orthogonal coordinates or frame fields, called tetrads or vierbeins in four dimensions. The formalism also introduces a local Lorentz connection called the spin connection. The connection is considered to be an independent field as in the Palatini formalism. When gravity is coupled to only non-fermionic fields, this formalism reduces to the usual metric formalism of general relativity on-shell, where the spin connection can be written in terms of the tetrads and their derivatives. If there are fermionic fields contributing to the stress-energy tensor, the spin connection has torsion components and remains independent.

This formalism is particularly useful for writing a Lagrangian for fermionic fields on curved space-time [5–7], as it highlights the spin connection as being analogous to a gauge field. In addition, the VEP formalism serves as the link between general relativity and BF theories of gravity [8, 9]. In this paper we investigate conformal transformations of vierbeins and spin connections. The motivation for this investigation is twofold. First, if the VEP action is exactly equivalent to the second-order Einstein-Hilbert action of gravity, all matter fields should couple to gravity 'in the same way' in both formulations. More precisely, the corresponding stress-energy tensors for matter should be equivalent in the two formulations, and symmetries of the matter fields should also be present in both the formulations. This is a trivial issue for minimally coupled matter fields, not so for non-minimally coupled fields, such as the conformally coupled scalar, which we investigate here. The other motivation is to study the conformal transformation of spin connection, which is useful in studying the conformal properties of fermions propagating on a curved background.

Conformal transformations were introduced by Weyl in an attempt to unify electromagnetism and general relativity [15], and have been useful in studying various properties of curved spacetimes [32]. Conformal transformations have been widely used in studying asymptotic flatness and initial value problem [10, 16–19], propagation of massless fields on a gravitational background [20–28], exact solutions [29–31, 33–37] and other problems where scale-independence is fundamental to our understanding of the system. Conformal invariance is also important in the study of quantum field theory on curved spacetime [11–14].

A conformal transformation is the scaling of the spacetime metric $g_{\mu\nu}$ with a strictly

positive, smooth function Ω^2,

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}. \tag{1.1}$$

This transformation alters lengths of spacetime intervals, but preserves angles. The conformally transformed spacetime and the original one have the same causal structure. Since Ω is a function of spacetime, the transformation of metric affects different entities like the Christoffel symbols, Riemann tensor and hence the Einstein-Hilbert action. For gauge fields in four dimensions, the matter action remains invariant under conformal transformation, while for other kinds of matter fields like the scalar, the action needs to be modified. We will consider the scalar field in Sec. III. Conformal transformation of the metric transforms the symmetric Christoffel symbols as

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + \delta^{\alpha}_{(\mu}\nabla_{\nu)} \ln\Omega - g_{\mu\nu} g^{\alpha\beta}\nabla_\beta \ln\Omega\,, \tag{1.2}$$

where the symmetric combination is defined as $A_{(\alpha}B_{\beta)} = A_\alpha B_\beta + A_\beta B_\alpha$, and thus the Ricci scalar can be written as

$$\begin{aligned}\tilde{R} = \Omega^{-2}\{&R - 2(n-1) g^{\mu\nu}\nabla_\mu\nabla_\nu \ln\Omega \\ &- (n-1)(n-2)(\nabla_\mu \ln\Omega)(\nabla_\mu \ln\Omega)\}\,.\end{aligned} \tag{1.3}$$

This is the general formula in n space-time dimensions. We will be concerned with the case where $n = 4$.

In this paper we discuss the conformal transformation of the vierbein and the spin connection and investigate the conformal properties of the action in VEP formalism. In Sec II, we briefly discuss the vierbein-Einstein-Palatini formalism and investigate the conformal transformation of the vierbein and the spin connection. The transformation of the spin connection is postulated by comparing the VEP action with the equivalent action in the metric formalism. In Sec III we consider the conformal scalar field and discover how to define its action in the VEP formalism. We show how conformal invariance demands torsion to be strictly zero. In Sec IV we show how the presence of fermionic fields produces an additional term in the spin connection, ultimately leading to torsion and breaking the invariance of the conformal scalar.

II. CONFORMAL TRANSFORMATION IN VEP FORMALISM

In the VEP formalism a local Lorentz space (also called the 'internal space' in analogy with gauge theories), isomorphic to tangent space, is defined at each point of spacetime. The isomorphism is given by tetrads, e^I_μ, which are considered to be orthonormal,

$$g^{\mu\nu} e^I_\mu e^J_\nu = \eta^{IJ}, \tag{2.1}$$

The inverse of tetrads, also called co-tetrads, are written as e^μ_I and are given by

$$e^\mu_I e^I_\nu = \delta^\mu_\nu \,. \tag{2.2}$$

The spacetime indices, denoted by lower-case Greek letters, are raised and lowered by spacetime metric g , while the internal indices, denoted by Upper-case Latin letters are raised and lowered by the internal Minkowski metric, η. The determinant of tetrads ($|e|$) is same as the square root of the determinant of the spacetime metric ($\sqrt{-g}$). A connection, D, is defined such that its action on any smooth section is given as

$$D_\mu S^I = \partial_\mu S^I + A^I_{\mu J} S^J \,, \tag{2.3}$$

where, the vector potential, A^{IJ}_μ is also called the spin connection. The curvature of D is calculated as

$$F^{IJ}_{\mu\nu} = \partial_\mu A^{IJ}_\nu - \partial_\nu A^{IJ}_\mu + [A_\mu, A_\nu]^{IJ} \,. \tag{2.4}$$

It is easy to see that the spin connection is antisymmetric by noting that $D_\mu \eta^{IJ}$ must vanish. We define a set of Christoffel symbols as

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_I \partial_\mu e^I_\nu + A^I_{\mu J} e^J_\nu e^\alpha_I \,. \tag{2.5}$$

This leads to a metric-compatible connection ∇ , but as there is no symmetry of Γ in the lower indices, this connection is not torsion-free a priori. The Riemann tensor, Ricci tensor and Ricci scalar are calculated respectively as,

$$R^\rho{}_{\sigma\mu\nu} = F^I_{\mu\nu J} e^\rho_I e^J_\sigma \,, \tag{2.6}$$

$$R_{\sigma\nu} = F^I_{\mu\nu J} e^\mu_I e^J_\sigma \,, \tag{2.7}$$

$$R = F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \,. \tag{2.8}$$

We can now write the VEP action for gravity by replacing the Ricci scalar by Eq. (2.8) and the metric determinant by that of tetrads.

$$S[e,A]=\frac{1}{2\kappa}\int |e|d^4x\, F^{IJ}_{\mu\nu}e^{\mu}_{I}e^{\nu}_{J}\,. \tag{2.9}$$

Here $\kappa=8\pi G$. Variation of the action with tetrads produces the following equation.

$$2F^{IJ}_{\lambda\nu}e^{\lambda}_{I}-e^{J}_{\nu}F^{KL}_{\rho\sigma}e^{\rho}_{K}e^{\sigma}_{L}=0. \tag{2.10}$$

Contracting with $e_{\mu J}$, and using Eq. (2.7), we get the familiar form

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=0. \tag{2.11}$$

This equation would be the vacuum Einstein's equation if the connection is torsion-free. Next we vary the action with respect to the spin connection and obtain the equation

$$-e^{\alpha}_{K}(\partial_\mu e^{K}_{\alpha})e^{\mu}_{I}e^{\nu}_{J}-(\partial_\mu e^{\mu}_{I})e^{\nu}_{J}-e^{\mu}_{I}(\partial_\mu e^{\nu}_{J})+A^{K}_{\mu I}e^{\mu}_{K}e^{\nu}_{J}-A^{K}_{\mu I}e^{\mu}_{J}e^{\nu}_{K}=0. \tag{2.12}$$

Upon contractions in the free internal indices with tetrads and after some index manipulations, we get

$$\delta^{\nu}_{\lambda}(\Gamma^{\alpha}_{\alpha\rho}-\Gamma^{\alpha}_{\rho\alpha})+\delta^{\nu}_{\rho}(\Gamma^{\alpha}_{\lambda\alpha}-\Gamma^{\alpha}_{\alpha\lambda})+\Gamma^{\nu}_{\rho\lambda}-\Gamma^{\nu}_{\lambda\rho}=0\,, \tag{2.13}$$

This equation gives the torsion-free condition after eliminating a vanishing trace. We can thus identify Eq. (2.10) with the vacuum Einstein's equation. Eq. (2.13) can be solved for the spin connection in the absence of matter,

$$A^{IJ}_{\mu}=\frac{1}{2}e_{\mu K}\left(\Theta^{KIJ}-\Theta^{IJk}-\Theta^{JKI}\right)\,, \tag{2.14}$$

where

$$\Theta^{IJK}=e^{I}_{\nu}\left[e^{\mu J}\partial_\mu e^{K\nu}-e^{\mu K}\partial_\mu e^{\nu J}\right]. \tag{2.15}$$

In presence of matter, however, the absence of torsion is not guaranteed. There may be a non-vanishing torsion component of the spin connection which cannot be expressed in terms of the tetrads only. Including matter fields, the VEP action reads

$$S_{Total}=\frac{1}{\kappa}\int |e|d^4xF^{IJ}_{\mu\nu}e^{\mu}_{I}e^{\nu}_{J}+S_M, \tag{2.16}$$

where $S_M=\int |e|d^4x\mathcal{L}_M$ is the action for the matter. The total VEP equation, obtained by variation with respect to the tetrad, is then

$$F^{IJ}_{\alpha\mu}e^{\alpha}_{I}-\frac{1}{2}e^{J}_{\mu}F^{KL}_{\alpha\beta}e^{\alpha}_{K}e^{\beta}_{L}=\kappa T_{\mu\alpha}e^{\alpha J}\,, \tag{2.17}$$

where $T_{\mu\alpha}$ is the usual energy-momentum tensor for the matter. As before, we can contract this equation with the tetrad to obtain the familiar form, $G_{\mu\nu} = \kappa T_{\mu\nu}$. On the other hand, if the spin connection couples to matter, the right hand side of Eq. (2.12) will not vanish, and there may be non-zero torsion. This happens when a fermionic matter field is present. In presence of torsion, the spin connection cannot be expressed purely in terms of the vierbein and its derivatives. We note in passing that sometimes the expression in Eq. (2.14) is referred to as the spin connection, with torsion being added as a separate field. In this paper we have included torsion, where present, in the spin connection, so Eq. (2.14) is the expression in the absence of matter, or as we shall see below more generally, in the absence of torsion. Let us now consider the conformal transformation of the VEP variables. The transformation of $g_{\mu\nu}$ suggests that the tetrads should transform in the following manner,

$$\tilde{e}^I_\mu = \Omega e^I_\mu \,, \tag{2.18}$$

while the co-tetrads should transform as

$$\tilde{e}^\mu_I = \Omega^{-1} e^\mu_I \,. \tag{2.19}$$

The transformation of the $\Gamma^\alpha_{\mu\nu}$ in the VEP formalism can be obtained by defining $\tilde{\Gamma}$ in terms of the transformed tetrads and spin connection,

$$\begin{aligned}\tilde{\Gamma}^\alpha_{\mu\nu} &= \tilde{e}^\alpha_I \partial_\mu \tilde{e}^I_\nu + \tilde{A}^I_{\mu J} \tilde{e}^J_\nu \tilde{e}^\alpha_I \\ &= \delta^\alpha_\nu \partial_\mu (\ln \Omega) + e^\alpha_I \partial_\mu e^I_\nu + \tilde{A}^I_{\mu J} e^J_\nu e^\alpha_I \,. \end{aligned} \tag{2.20}$$

The transformation of $A^I_{\mu J}$ can be calculated from this by positing that the corresponding connection $\tilde{\nabla}$ is compatible with the transformed metric $\tilde{g}_{\mu\nu}$. Using Eq. (2.20) we can write

$$\begin{aligned}\tilde{\nabla}_\mu \tilde{g}_{\alpha\beta} &= \partial_\mu \tilde{g}_{\alpha\beta} - \tilde{\Gamma}^\nu_{\mu\alpha} \tilde{g}_{\nu\beta} - \tilde{\Gamma}^\nu_{\mu\beta} \tilde{g}_{\alpha\nu} \\ &= \partial_\mu (\Omega^2 g_{\alpha\beta}) - 2\Omega^2 \partial_\mu (\ln \Omega) g_{\alpha\beta} - \Omega^2 (\partial_\mu e_{I(\alpha}) e^I_{\beta)} - \Omega^2 \tilde{A}^{IJ}_\mu (e_{J(\alpha} e_{\beta)I}) \\ &= 0 \,. \end{aligned} \tag{2.21}$$

In the last equality, we have used the orthonormality of the tetrads and the antisymmetry of the spin connection. Quite clearly, metric compatibility is not sufficient to determine the transformation of A, and only shows antisymmetry in the internal indices $I\,, J$. We can find the transformation of the spin connection by using the conformal transformation of the

Christoffel symbols and using Eq. (2.5) [38]

$$\tilde{A}^I_{\mu J} = A^I_{\mu J} + e^I_\mu e_{\nu J}\nabla^\nu \ln\Omega - e_{\mu J} e^I_\nu \nabla^\nu \ln\Omega\,. \tag{2.22}$$

This leads to the following transformation of the Ricci scalar

$$\tilde{R} = \Omega^{-2}\left[F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J - 6\nabla_\alpha\nabla^\alpha \ln\Omega - 6 g_{\alpha\beta}\nabla^\alpha \ln\Omega \nabla^\beta \ln\Omega + 2\left(\Gamma^\mu_{\mu\alpha} - \Gamma^\mu_{\alpha\mu}\right)\nabla^\alpha \ln\Omega\right]\,. \tag{2.23}$$

We have written the last term using the Christoffel symbols Γ because we need to talk about its symmetry. We note that the above equation implies that the transformed R is same as that in Eq. (1.3) in four dimensions if $\Gamma^\mu_{\alpha\nu}$ is symmetric in the lower indices. It is worth noting that presence of the torsion term also alters the invariance of Weyl tensor and it has interesting aspects in VEP formalism. We are not going to discuss those here and will be looked at elsewhere. We cannot however assume this symmetry of Γ, until using the equation of motion obtained by variation of the spin connection. We denote the torsion tensor as

$$C^\alpha{}_{\mu\nu} = \Gamma^\alpha_{[\mu\nu]}\,. \tag{2.24}$$

The VEP action is written in terms of the tetrads rather than the metric, so we write the conformally transformed VEP action as

$$\begin{aligned}\tilde{S}(e, A, \Omega) =& \frac{1}{\kappa}\int \Omega^2 \left[F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J - 6 e^\mu_I e^{\nu I}\nabla_\mu \partial_\nu \ln\Omega\right.\\ &\left.-6 e^\mu_I e^{\nu I}\left(\partial_\mu \ln\Omega\right)\left(\partial_\nu \ln\Omega\right) + 2C^\mu{}_{\alpha\mu}\nabla^\alpha \ln\Omega\right]|e| d^4x\,.\end{aligned} \tag{2.25}$$

Here the covariant derivative is to be understood as being written in terms of A,

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - e^\alpha_I \partial_\mu e^I_\nu V_\alpha + A^I_{\mu J} e^J_\nu e^\alpha_I V_\alpha\,. \tag{2.26}$$

While arriving at the transformation of the Ricci scalar and the action, we did not find the transformation of torsion. The transformation formula given in Eq. (2.22) does not tell us whether or not the spin connection includes torsion. However, as we shall see below, the presence of fermionic fields guarantees that the spin connection will contain torsion. The transformation of the torsion field will be evident in that situation.

III. CONFORMALLY COUPLED SCALAR FIELD

Now that we have established the conformal transformations of the tetrad and spin connection, we can investigate the behavior of scalar fields conformally coupled to gravity.

Before doing this in the vierbein-Einstein-Palatini formalism, let us quickly go over the treatment of the conformal scalar in the metric formalism. The equation of conformal scalar is given by

$$\nabla_\mu \nabla^\mu \phi - \frac{1}{6} R\phi = 0\,, \tag{3.1}$$

which can be obtained from the matter action

$$S(\phi, g) = -\int \sqrt{-g}\, d^4x \left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{12} R\phi^2\right]\,. \tag{3.2}$$

This matter action is invariant under the conformal transformation of Eq. (1.1) provided the scalar field transforms as

$$\phi \to \Omega^{-1} \phi\,. \tag{3.3}$$

Variation of the matter action with respect to the metric produces the energy-momentum tensor corresponding to the conformal scalar field, which now includes a part that depends on the geometry because of the $R\phi^2$ term,

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi + \frac{1}{6} G_{\mu\nu} \phi^2 + \frac{1}{6} \left[g_{\mu\nu} \nabla_\sigma \nabla^\sigma \phi^2 - \nabla_\mu \nabla_\nu \phi^2\right]\,. \tag{3.4}$$

This $T_{\mu\nu}$ is a conserved tensor as expected,

$$\nabla^\mu T_{\mu\nu} = 0\,. \tag{3.5}$$

Let us now take a look at the conformal scalar in the VEP formalism. We will investigate two cases – with and without fermionic fields. Writing the Ricci scalar in terms of the tetrad and the spin connection, and including the fermion action [5–7], we can write the total action with conformal scalar and fermion matter sources,

$$\begin{aligned} S[e, A, \phi, \psi] = \frac{1}{2\kappa} \int |e| d^4x\, F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J + \int |e| d^4x \left[-\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{12} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi^2\right] \\ + \int |e| d^4x \left[\bar{\psi} \gamma^k e^\mu_k \left(\partial_\mu + \frac{i}{4} A^{IJ}_\mu \sigma_{IJ}\right) \psi\right]. \end{aligned} \tag{3.6}$$

We have considered massless fermions here since a mass term will break conformal invariance. Let us first see what happens in the absence of fermionic fields. Variation of the action with respect to the scalar field produces the equation

$$\nabla_\mu \nabla^\mu \phi - \frac{1}{6} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi = 0\,. \tag{3.7}$$

Using the transformation of the Ricci scalar from Eq. (2.23), we see that the scalar equation Eq. (3.7) transforms as

$$\Omega^{-3}\left[\nabla_\mu\nabla^\mu\phi - \frac{1}{6}F^{IJ}_{\mu\nu}e^\mu_I e^\nu_J\phi - \frac{1}{3}C^\mu{}_{\sigma\mu}\nabla^\sigma\phi\right] = 0\,. \tag{3.8}$$

There is thus an extra term that apparently breaks the covariance and this term is a direct interaction of scalar field and torsion. If the spacetime is torsion-free, we get back the covariant equation. However, there is still another problem with the scalar action that is crucial to understanding what happens in the presence of fermions, as we shall see now.

Let us extremise the total action, Eq. (3.6), still without fermions, with respect to tetrads. This produces Einstein's equation, in the form

$$\begin{aligned} F^{IJ}_{\mu\nu}e^\mu_I - \frac{1}{2}e^J_\nu F^{KL}_{\alpha\beta}e^\alpha_K e^\beta_L = \kappa\Big[&\eta^{IJ}e^\mu_I\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{6}F^{IJ}_{\mu\nu}e^\mu_I\phi^2 \\ &-\frac{1}{2}\eta^{KL}e^\alpha_K e^\beta_L e^J_\nu\nabla_\alpha\phi\nabla_\beta\phi + \frac{1}{12}e^J_\nu e^\alpha_K e^\beta_L F^{KL}_{\alpha\beta}\phi^2\Big]\,. \end{aligned} \tag{3.9}$$

In order to convert this equation to the usual form we contract both sides by $e_{J\lambda}$, to obtain

$$G_{\mu\nu} = \kappa\left[\nabla_\nu\phi\nabla_\lambda\phi - \frac{1}{2}g_{\nu\lambda}\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{6}G_{\mu\nu}\phi^2\right]\,. \tag{3.10}$$

The right hand side of the above equation should contain the energy-momentum tensor, but upon comparing with Eq. (3.4), we see that the expression is incorrect as it stands, and it will not be a conserved tensor. This is because there is an obvious flaw in the way we have written the scalar action in VEP formalism, specifically in the non-minimal term $R\phi^2$. The connection used in the VEP formalism is not torsion-free a priori, but the torsion vanishes on shell. However, the $R\phi^2$ term that contributes to the stress-energy tensor in Eq. (3.4) is constructed from a torsion-free connection. Therefore, in the VEP formalism we need to construct the non-minimal $R\phi^2$ term only from the torsion-less part of the connection. To do this, let us start with a Christoffel symbol Γ corresponding to a general metric compatible connection with torsion,

$$\Gamma^\alpha_{\mu\nu} = \widehat{\Gamma}^\alpha_{\mu\nu} - \frac{1}{2}\left[C_{\mu\nu}{}^\alpha + C_{\nu\mu}{}^\alpha - C^\alpha{}_{\mu\nu}\right]\,, \tag{3.11}$$

where $\widehat{\Gamma}$ is the torsion-free part and the extra part is termed as contorsion tensor. The Ricci scalar corresponding to Γ is calculated to be

$$R = \widehat{R} - 2g^{\nu\sigma}\widehat{\nabla}_\nu C^\mu{}_{\mu\sigma} - C^\mu{}_{\mu\sigma}C^\nu{}_\nu{}^\sigma + \frac{1}{4}C^{\mu\nu\sigma}C_{\mu\nu\sigma} + \frac{1}{2}C^{\mu\nu\sigma}C_{\nu\mu\sigma}\,. \tag{3.12}$$

Here $\widehat{R}$ and $\widehat{\nabla}$ are respectively the Ricci scalar and connection corresponding to the torsion-free $\widehat{\Gamma}$. Now it is easy to determine the necessary modification of the coupling of the conformal scalar to gravity. As we mentioned above, the Ricci scalar that appears in the $R\phi^2$ term in Eq. (3.2) is actually $\widehat{R}$, corresponding to the torsion-free connection. When going over to the VEP formalism, we note that $F^{IJ}_{\mu\nu}e^\mu_I e^\nu_J$ corresponds to R, so we should write the total VEP action as

$$\begin{aligned} S[e, A, \phi] = \int |e| d^4x F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \left(\frac{1}{2\kappa} - \frac{\phi^2}{12}\right) \\ - \int |e| d^4x \left[\frac{1}{2} e^\mu_I e^{\nu I} \partial_\mu\phi\partial_\nu\phi - \frac{1}{6} e^\nu_I e^{\sigma I} C^\mu{}_{\mu\sigma}\partial_\nu\phi^2 \right. \\ \left. + \frac{1}{12}\left(C^\mu{}_{\mu\sigma}C^\nu{}_\nu{}^\sigma - \frac{1}{4}C^{\mu\nu\sigma}C_{\mu\nu\sigma} - \frac{1}{2}C^{\mu\nu\sigma}C_{\nu\mu\sigma}\right)\phi^2\right], \end{aligned} \tag{3.13}$$

where the torsion tensor is to be understood as

$$C^\alpha{}_{\mu\nu} = e^\alpha_I \partial_\mu e^I_\nu - e^\alpha_I \partial_\nu e^I_\mu + A^I_{\mu J} e^J_\nu e^\alpha_I - A^I_{\nu J} e^J_\mu e^\alpha_I \,, \tag{3.14}$$

which form can be found by combining Eq. (2.5) with Eq. (2.24). We have written the action by integrating by parts, noting that torsion does not affect metric-compatibility of the connection, so $\widehat{\nabla}$ is metric compatible and thus annihilates the tetrad. Let us first find out the scalar equation by extremising the above action with respect to ϕ.

$$\begin{aligned} \nabla_\mu \nabla^\mu \phi - \frac{1}{6} F^{IJ}_{\mu\nu} e^\mu_I e^\nu_J \phi - \frac{1}{3}\nabla^\nu C^\mu_{\mu\nu}\phi \\ + \frac{1}{6}\left(C^\mu{}_{\mu\sigma}C^\nu{}_\nu{}^\sigma - \frac{1}{4}C^{\mu\nu\sigma}C_{\mu\nu\sigma} - \frac{1}{2}C^{\mu\nu\sigma}C_{\nu\mu\sigma}\right)\phi = 0\,. \end{aligned} \tag{3.15}$$

To see the transformation of this equation, we need to know the transformation of the torsion tensor explicitly. However, as we have not considered fermionic field, we expect torsion to vanish on shell, i.e. when equations of motion are satisfied. As we will see below, the equations of motion corresponding to the spin connection imply a vanishing torsion.

Before we get to that, let us calculate the energy momentum tensor $T_{\mu\nu}$ of the scalar field. As we have already seen, $T_{\mu\nu}$ appears on the right hand side of the Einstein equation obtained by extremising the action with tetrads. The term in the action of Eq. (3.13) that is of particular interest is the one of first order in C. The remaining terms which depend on the torsion do so quadratically, and will contribute in the energy-momentum tensor at

least linearly in torsion. Thus their contribution to $T_{\mu\nu}$ will be zero eventually upon using the equation of motion for the spin connection, as we shall see below.

Variation of the above action with respect to the tetrad produces

$$F^{IJ}_{\mu\nu}e^\mu_I - \frac{1}{2}e^J_\nu F^{KL}_{\alpha\beta} = \kappa\Big(\eta^{IJ}e^\mu_I\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}e^J_\nu\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{6}\big(F^{IJ}_{\mu\nu}e^\mu_I - \frac{1}{2}e^J_\nu F^{KL}_{\alpha\beta}\big)\phi^2$$
$$-\frac{1}{6}\Big[e^J_\alpha\big(\delta^\alpha_\nu\nabla_\sigma\nabla^\sigma\phi^2 - \nabla^\alpha\nabla_\nu\phi^2\big)\Big] + f(C)\Big)\,, \qquad (3.16)$$

where $f(C)$ contains all the terms which contain torsion explicitly. In order to compare with the usual form, we contract the above equation with the tetrad. After collecting terms and simplifying, we find

$$G_{\mu\nu} = \kappa\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi + \frac{1}{6}G_{\mu\nu}\phi^2 + \frac{1}{6}\left(g_{\mu\nu}\nabla_\alpha\nabla^\alpha\phi^2 - \nabla_\mu\nabla_\nu\phi^2\right) + f(C)\right]. \qquad (3.17)$$

We see that we have recovered Einstein equations for gravity coupled to a conformal scalar field, apart from a term which depends on torsion. Let us now vary the action with respect to the spin connection, we can expect that a condition on the torsion will emerge. The equation of motion that we get as a result of the variation is

$$2\left(\tfrac{1}{2\kappa} - \tfrac{\phi^2}{12}\right)\left[-e^\alpha_K(\partial_\mu e^K_\alpha)e^\mu_I e^\nu_J - (\partial_\mu e^\mu_I)e^\nu_J - e^\mu_I(\partial_\mu e^\nu_J) + A^K_{\mu I}e^\mu_K e^\nu_J - A^K_{\mu I}e^\mu_J e^\nu_K\right]$$
$$-\frac{1}{6}e^\nu_I e^\sigma_J C^\mu{}_{\mu\sigma}\phi^2 + \frac{1}{12}C^{\mu\nu\sigma}e_{\mu I}e_{\sigma J}\phi^2 + \frac{1}{12}(C^{\nu\mu\sigma} - C^{\sigma\mu\nu})e_{\mu I}e_{\sigma J}\phi^2 = 0\,. \qquad (3.18)$$

We now contract the equation with $e^I_\rho e^J_\lambda$, and then antisymmetrise in ρ, λ, to produce

$$\frac{1}{\kappa}\left[\delta^\nu{}_\lambda C^\mu{}_{\mu\rho} - \delta^\nu{}_\rho C^\mu{}_{\mu\lambda} + C^\nu{}_{\rho\lambda}\right] = 0\,. \qquad (3.19)$$

We note that the terms containing the scalar field have canceled one another and we are left with the same equation as Eq. (2.13), which produces the torsion-free condition $C^\rho{}_{\mu\nu} = 0$. This makes the terms explicitly dependent on torsion vanish, and also enables us to identify ∇ with $\widehat{\nabla}$. The scalar equation Eq. (3.15) thus reads

$$\widehat{\nabla}_\mu\widehat{\nabla}^\mu\phi - \frac{1}{6}\widehat{R}\phi = 0\,, \qquad (3.20)$$

where, as before, the hat corresponds to torsion-free quantities. This is the same as Eq. (3.1), and is therefore conformally invariant. We note however that, if fermionic matter is present as well, torsion does not vanish, and the stress-energy tensor for the conformal scalar field then has to include terms dependent on the torsion and the invariance of the equation is no longer there as we will see in the following section.

IV. FERMION, TORSION, AND THEIR EFFECT ON THE CONFORMAL SCALAR

In the absence of fermionic matter, it is always possible to solve for the components of the spin connection in terms of the tetrads, and the corresponding connection is torsion-free. In such cases conformal transformations do not pose any problem. However, as we shall see in this section, the presence of fermions alter the behaviour of the spin connection under conformal transformations, which in turn affect the covariance of the conformal scalar field.

Let us begin from the total action given in Eq. (3.13), including the term for fermionic fields. The Dirac equation can be obtained by extremising the action with respect to $\bar{\psi}$,

$$\gamma^K e^\mu_K \left(\partial_\mu + \frac{i}{4} A^{IJ}_\mu \sigma_{IJ}\right)\psi = 0\,. \tag{4.1}$$

The γ and σ matrices carry internal indices and have usual properties with metric signature $(-+++)$. We note here that for the choice of signature $(+---)$, which is popular in quantum field theory, we need to change $\gamma \to i\gamma$. Unlike in the case of bosonic matter fields, the spin connection couples directly to fermions, and its components contribute to torsion in the presence of fermions. In order to see this, let us find the equation by varying the action with respect to the spin connection,

$$\begin{aligned} 2\left(\tfrac{1}{2\kappa} - \tfrac{\phi^2}{12}\right) &\left[-e^\alpha_K(\partial_\mu e^K_\alpha)e^\mu_I e^\nu_J - (\partial_\mu e^\mu_I)e^\nu_J - e^\mu_I(\partial_\mu e^\nu_J) + A^K_{\mu I} e^\mu_K e^\nu_J - A^K_{\mu I} e^\mu_J e^\nu_K\right] \\ &- \frac{1}{6} e^\nu_I e^\sigma_J C^\mu{}_{\mu\sigma}\phi^2 + \frac{1}{12} C^{\mu\nu\sigma} e_{\mu I} e_{\sigma J}\phi^2 + \frac{1}{12}(C^{\nu\mu\sigma} - C^{\sigma\mu\nu}) e_{\mu I} e_{\sigma J}\phi^2 \\ &= \frac{i}{4}\bar{\psi}\gamma^K e^\nu_K \sigma_{IJ}\psi\,. \end{aligned} \tag{4.2}$$

Upon some suitable contractions and cyclic permutations, we see that the terms containing the scalar field cancel one another as in the previous section, and we are left with the following expression of the spin connection:

$$\begin{aligned} A^{IJ}_\mu = \frac{1}{2} e_{\mu K}\left(\Theta^{KIJ} - \Theta^{IJk} - \Theta^{JKI}\right) &+ \frac{i\kappa}{4}\bar{\psi}(\gamma^I\sigma^{JL} + \gamma^J\sigma^{LI} - \gamma^L\sigma^{IJ}) e_{\mu L}\psi \\ &+ \frac{i\kappa}{4}\bar{\psi}(e^I_\mu \gamma_K \sigma^{JK} - e^J_\mu \gamma_K \sigma^{IK})\psi\,. \end{aligned} \tag{4.3}$$

Comparing the above expression with Eq. (2.14), we see that some new terms have appeared due to the fermionic field. The first term in the equation above is the torsion-free part, but the last two terms contribute to torsion. In fact, these two terms result in the following non-vanishing expression for the torsion tensor:

$$C^\nu{}_{\rho\lambda} = \frac{i\kappa}{2}\bar{\psi}\gamma^K e^\nu_K e^I_\rho e^J_\lambda \sigma_{IJ}\psi + \frac{i\kappa}{4}\bar{\psi}\gamma^J\left(\delta^\nu_\rho e^I_\lambda - \delta^\nu_\lambda e^I_\rho\right)\sigma_{IJ}\psi\,. \tag{4.4}$$

It is interesting to note that in the absence of the torsion-dependent terms in the action of Eq. (3.13), the scalar field would have appeared in the expressions for the spin connection and torsion.

Let us consider the conformal transformation of massless fermionic field. We take the conformal weight of fermion to be w. Using the transformation of tetrads and spin connection, Eq. (2.18) and Eq. (2.22) respectively, we see that the fermionic terms in the action transform as

$$S_F \to \int |e| d^4x\, \Omega^{3+2w} \bar{\psi}\gamma^K e^\mu_K \left[\partial_\mu + \tfrac{i}{4} A^{IJ}_\mu \sigma_{IJ} + \left(w + \tfrac{3}{2}\right) \partial_\mu \ln\Omega\right] \psi\,. \tag{4.5}$$

Evidently, $w = -\frac{3}{2}$ will keep this part of the action invariant, i.e., the fermionic field should transform as

$$\psi, \bar{\psi} \to \Omega^{-\frac{3}{2}}\psi, \Omega^{-\frac{3}{2}}\bar{\psi}\,. \tag{4.6}$$

The invariance of the action also implies that the Dirac equation, Eq. (4.1), is conformally covariant. This result is in conformation with previous results [38]. However, there is something else we can do. Suppose we take the on-shell expression of the spin connection as given in Eq. (4.3) and insert it into the Dirac equation. The result is a non-linear equation [40], and there appears to be no way to make the equation invariant or covariant. The on-shell expressions not only make the Dirac equation non-linear, they also break the covariance of the conformal scalar. As we can see from Eq. (4.3), there are two parts of the spin connection. The conformal transformation of the torsion-free part can be found by combining Eq. (2.14) with Eq. (2.22),

$$\frac{1}{2} e_{\mu K}\left(\Theta^{KIJ} - \Theta^{IJk} - \Theta^{JKI}\right) \quad\to\quad \frac{1}{2} e_{\mu K}\left(\Theta^{KIJ} - \Theta^{IJk} - \Theta^{JKI}\right) + \left(e^I_\mu e^{\nu J} - e^J_\mu e^{\nu I}\right)\partial_\nu \ln\Omega\,. \tag{4.7}$$

However, using the transformation of $\bar{\psi}, \psi$, we see that the part of the spin connection that contributes to torsion transforms as

$$\begin{aligned} & \frac{i\kappa}{4}\bar{\psi}\left(\gamma^I\sigma^{JL} + \gamma^J\sigma^{LI} - \gamma^L\sigma^{IJ}\right) e_{\mu L}\psi + \frac{i\kappa}{4}\bar{\psi}\left(e^I_\mu \gamma_K \sigma^{JK} - e^J_\mu \gamma_K \sigma^{IK}\right)\psi \\ \to\quad & \Omega^{-2}\left(\frac{i\kappa}{4}\bar{\psi}\left(\gamma^I\sigma^{JL} + \gamma^J\sigma^{LI} - \gamma^L\sigma^{IJ}\right) e_{\mu L}\psi + \frac{i\kappa}{4}\bar{\psi}\left(e^I_\mu \gamma_K \sigma^{JK} - e^J_\mu \gamma_K \sigma^{IK}\right)\psi\right)\,. \end{aligned} \tag{4.8}$$

So, there is a relative conformal weight between the two parts. Thus we cannot write the transformation of the total connection in the form of original plus variation. Also, torsion as given by Eq. (4.4) transforms homogeneously with overall conformal weight -2,

$$C^\nu{}_{\rho\lambda} \to \Omega^{-2} C^\nu{}_{\rho\lambda}\,. \tag{4.9}$$

It is thus clear that torsion tensor (with index positions as above) does have a relative conformal weight over the torsion-free part. In order to see the effect of torsion on the scalar field, we consider Eq. (3.15), which is the equation of motion for the conformal scalar field. The first thing to note is that, using the on-shell expression Eq. (4.3) for the spin connection, and the expression Eq. (4.4) for the torsion components, we can reduce Eq. (3.15) to the form

$$\nabla_\mu \nabla^\mu \phi - \frac{1}{6}\widehat{R}\phi = 0\,. \tag{4.10}$$

We do not see the linear or quadratic torsion terms in this equation in contrast to Eq. (3.15). The reason is obvious from Eq. (3.12). If we remove the $\nabla_\mu \nabla^\mu \phi$ from Eq. (3.15), the remaining terms add up to $\widehat{R}\phi^2$, which does not have any contribution from torsion in it.

However, the above equation still has a contribution of torsion coming from the kinetic part. We can separate out the contorsion part from the derivatives to write Eq. (4.10) as

$$\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6}\widehat{R}\phi + C^\mu{}_\mu{}^\alpha \partial_\alpha \phi = 0\,. \tag{4.11}$$

Now, using the on-shell expression of C, we can write Eq. (4.11) as

$$\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6}\widehat{R}\phi + \frac{i\kappa}{4}\bar{\psi}\gamma^J e^{I\alpha}\sigma_{IJ}\psi\partial_\alpha \phi = 0\,. \tag{4.12}$$

The last term on the left hand side of this equation is an interaction term between the scalar field and the fermion, and this term is not conformally covariant as we will see now. Considering the transformations of the various quantities, we find that the above equation conformally transforms as

$$\Omega^{-3}\left(\widehat{\nabla}_\mu \widehat{\nabla}^\mu \phi - \frac{1}{6}\widehat{R}\phi\right) + \frac{i\kappa}{4}\Omega^{-5}\left(\bar{\psi}\gamma_L\sigma^{LI}\psi e_I^\alpha \partial_\alpha \phi - \bar{\psi}\gamma_L\sigma^{LI}\psi e_I^\alpha \phi \partial_\alpha \ln\Omega\right) = 0\,. \tag{4.13}$$

The first thing to notice about this equation is that it is no longer invariant. This result is in contradiction to a previous work [38], where it was claimed that the scalar field can be made invariant even in the presence of fermionic fields. However, torsion components were ignored in [38] when considering conformal transformations. It was claimed that there are two choices: either the spin connection and tetrads are treated as independent variables, or the tetrads are treated as the only independent variables. Only the latter option was considered and the spin connection was written in terms of tetrads both on- and off-shell.

But as we have found (and also know from earlier work [6, 39]), these two are not real choices – one cannot simply assume the torsion-free condition a priori, and the tetrads and

the spin connection must be treated as independent variables. Secondly, as we have found, the terms that break the conformal invariance of the scalar field equation are interactions between the scalar and fermionic fields. These terms have a different conformal weight from the purely bosonic part of the equation, and in addition there is a term containing the derivative of Ω. As far as we understand, these terms cannot be compensated by the transformation of any known interaction so as to make the equation invariant.

Thus, it is not possible to write a conformally invariant scalar equation in the presence of fermionic fields.

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